
OPTΣX
Mathematical
Modeling System

OPTEX Mathematical Modeling System - Chief Scientist DecisionWare International Corp. (OPTEX MMS 374838-456059)

**TUTORIAL**

DO Analytics

# VEHICLE ROUTING PROBLEM

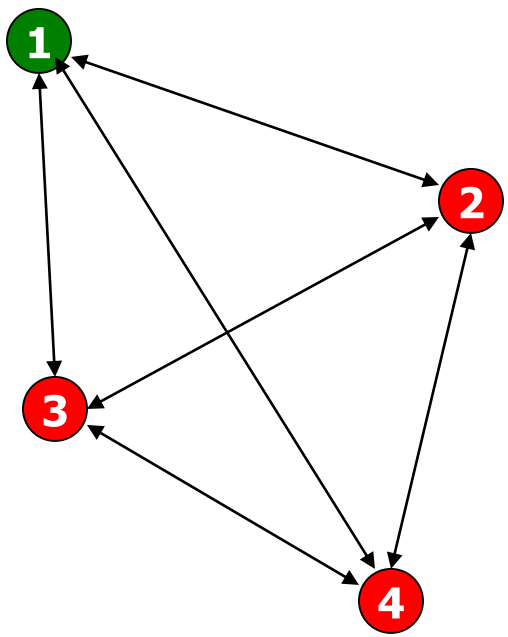
**Powered by**

**Think the mathematical model and OPTEX will make the software for you**

Mathematical Modeling System

Aceptar | Usuario: VRP | Clave | Cambiar Clave | Nueva Clave | Cancelar

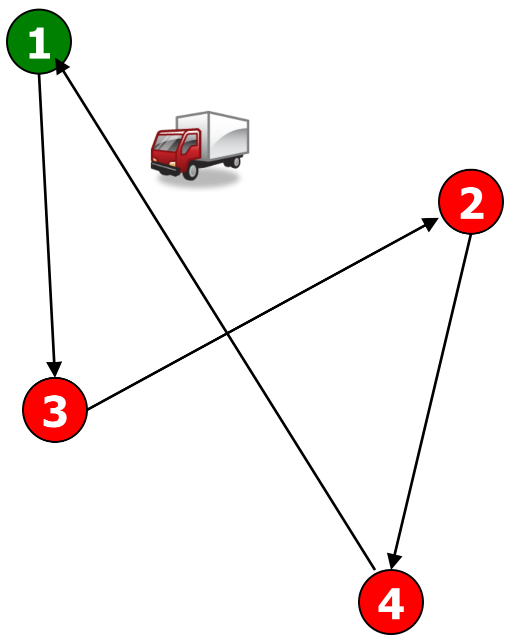
# SET COVERING
# TSP: TRAVEL SALESMAN PROBLEM




**Choose the optimal sequence that minimizes the costs of visiting all the nodes that make up a path, starting from a default source**

Analytics

# SET COVERING
# TSP: TRAVEL SALESMAN PROBLEM




$$Min \sum_i \sum_j c_{ij} x_{ij}$$

*subjet to*

$$\sum_j x_{ij} = 1 \quad \forall i$$

$$\sum_j x_{ji} = 1 \quad \forall i$$

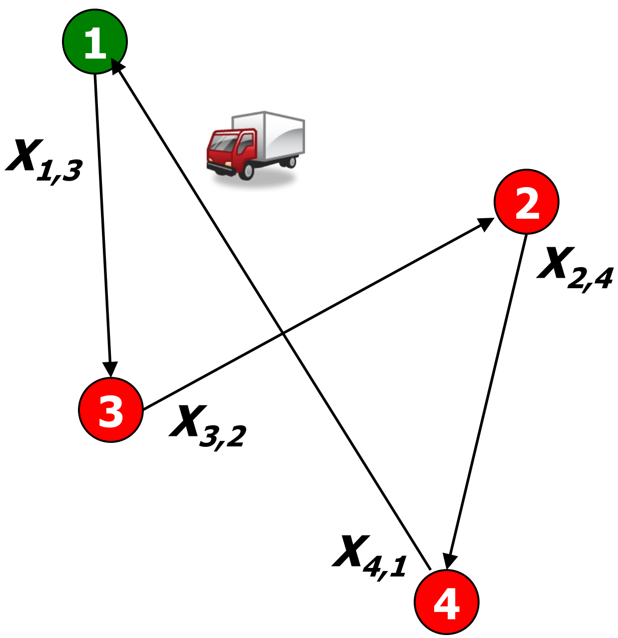
$$x_{ij} \in \{0,1\}$$

$$x_{ii} = 0$$

$c_{ij}$ *cost of going from i to j*

$x_{ij}$ *decision of going from i to j*

# SET COVERING
# TSP: TRAVEL SALESMAN PROBLEM




$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ij}$$

**subjet to**

$$\Sigma_j\, x_{ij} = 1 \quad \forall i$$

$$\Sigma_j\, x_{ji} = 1 \quad \forall i$$

$$x_{ij} \in \{0,1\}$$

$$x_{ii} = 0$$

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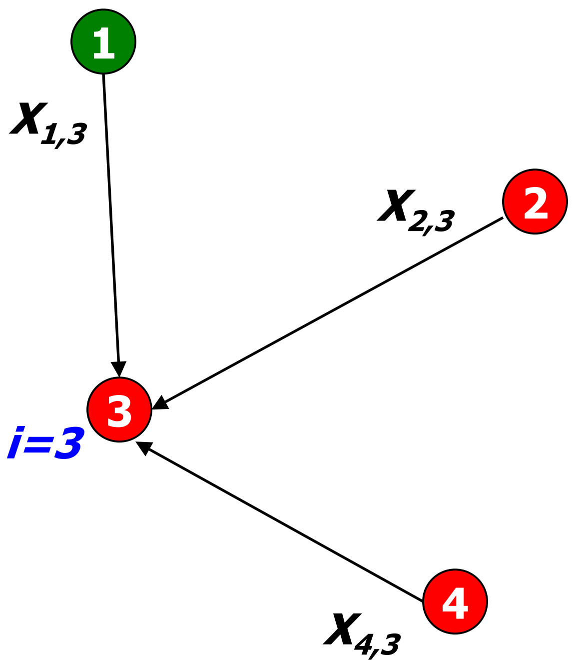


*Input balance equation*

$$\text{Min } \Sigma_i \Sigma_j \; c_{ij} \, x_{ij}$$

subjet to

$$\Sigma_j \, x_{ij} = 1 \quad \forall i$$

$$\Sigma_j \, x_{ji} = 1 \quad \forall i$$

$$x_{ij} \in \{0,1\}$$

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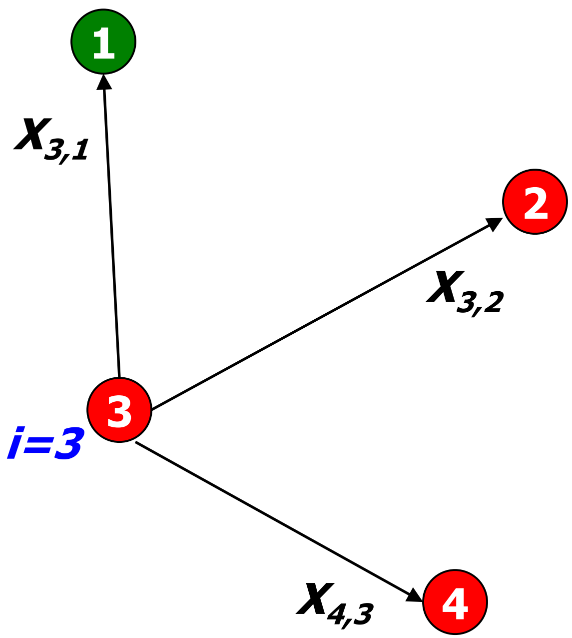
# SET COVERING
# TSP: TRAVEL SALESMAN PROBLEM




***Output balance equation***

$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ij}$$

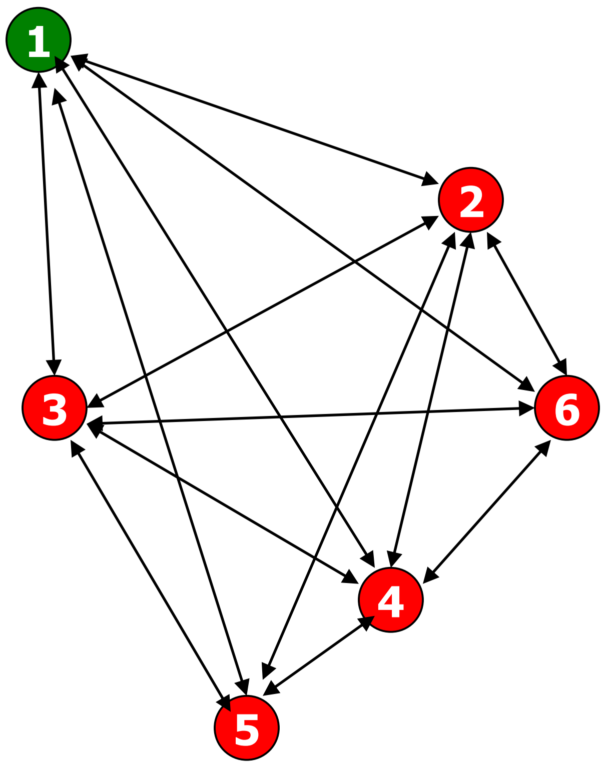
*subjet to*

$$\Sigma_j\, x_{ij} = 1 \quad \forall i$$
$$\Sigma_j\, x_{ji} = 1 \quad \forall i$$
$$x_{ij} \in \{0,1\}$$
$$x_{ii} = 0$$

$c_{ij}$ *cost of going from i to j*

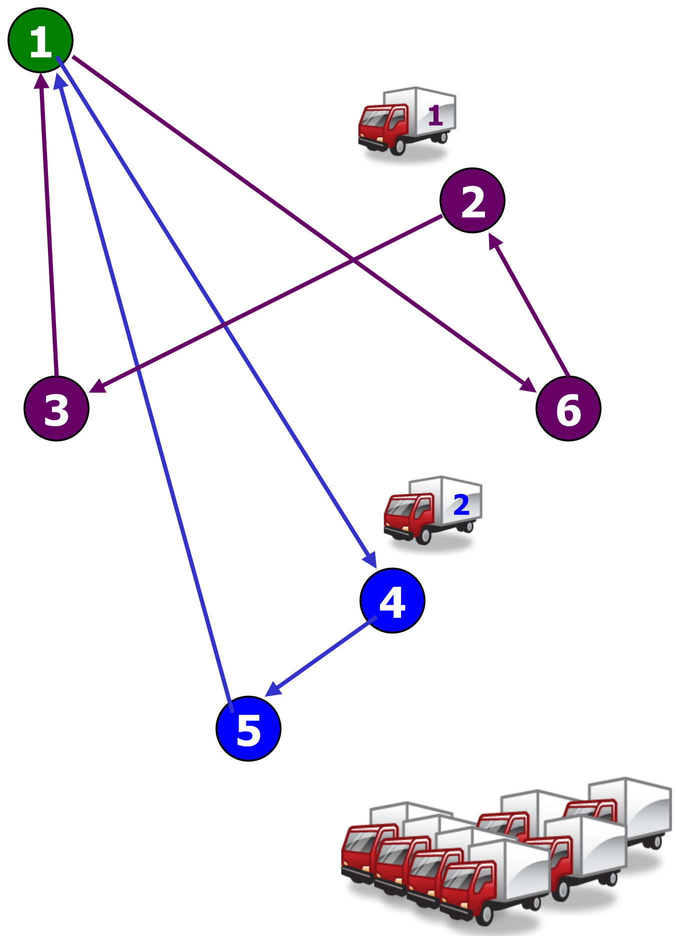
$x_{ij}$ ***decision of going from i to j***

# VRP: VEHICLE ROUTING PROBLEM




**The problem is to determine the nodes that must integrate the different routes that minimize the costs of visiting all the nodes of a distribution/recollection system, starting from a default source , using a fleet of homogenous vehicles.**

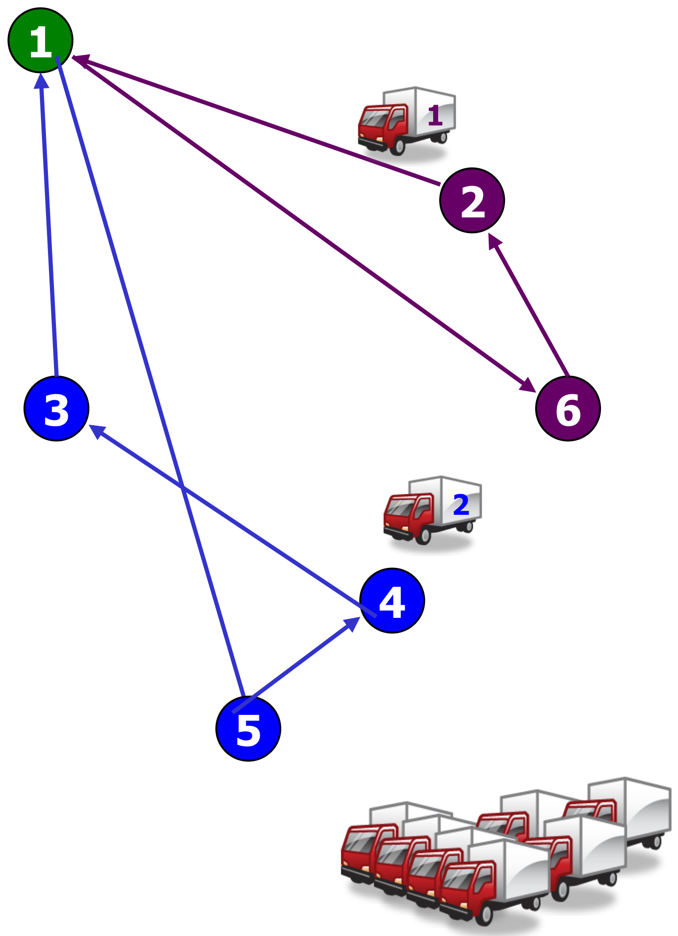
# VRP: VEHICLE ROUTING PROBLEM




1. You must select the set of nodes that make up the route/path

2. You must select the sequence of nodes within the route (TSP)

# VRP: VEHICLE ROUTING PROBLEM

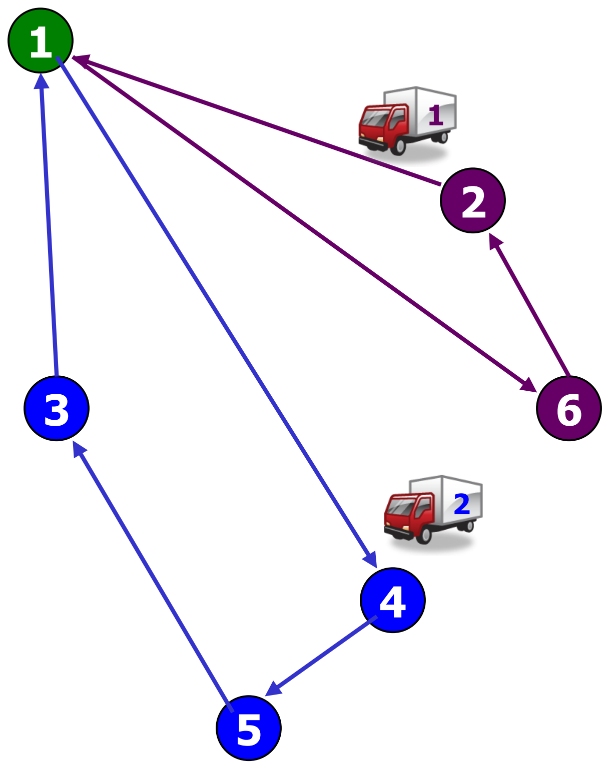



1. **You must select the set of nodes that make up the route/path**

2. **You must select the sequence of nodes within the route**

   **(TSP)**

# VRP: VEHICLE ROUTING PROBLEM




$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv} + \Sigma_v\, d_v y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v\, x_{ijv} = 1 \quad \forall i \neq 1$$

$$\Sigma_j\, x_{jiv} = \Sigma_j\, x_{jiv} \quad \forall i \ \ \forall v$$

$$\Sigma_i \Sigma_j\, x_{ijv} \leq y_v \quad \forall v$$

$$y_v \in \{0,1\},\ x_{ijv} \in \{0,1\}$$

$$x_{iiv} = 0$$

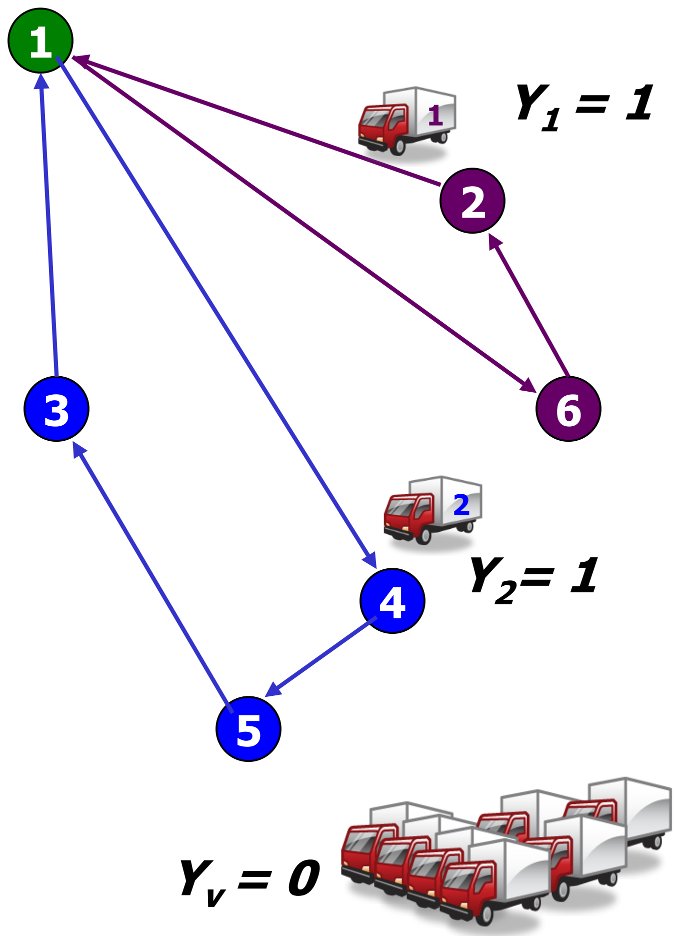
$d_v$ Cost activate route v

$c_{ijv}$ Cost of going from i to j using the route v

$y_v$ Decision to activate the route v

$x_{ijrv}$ Decision to go from i to j using route v

# VRP: VEHICLE ROUTING PROBLEM




$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv} + \Sigma_v\, d_v\, y_v$$

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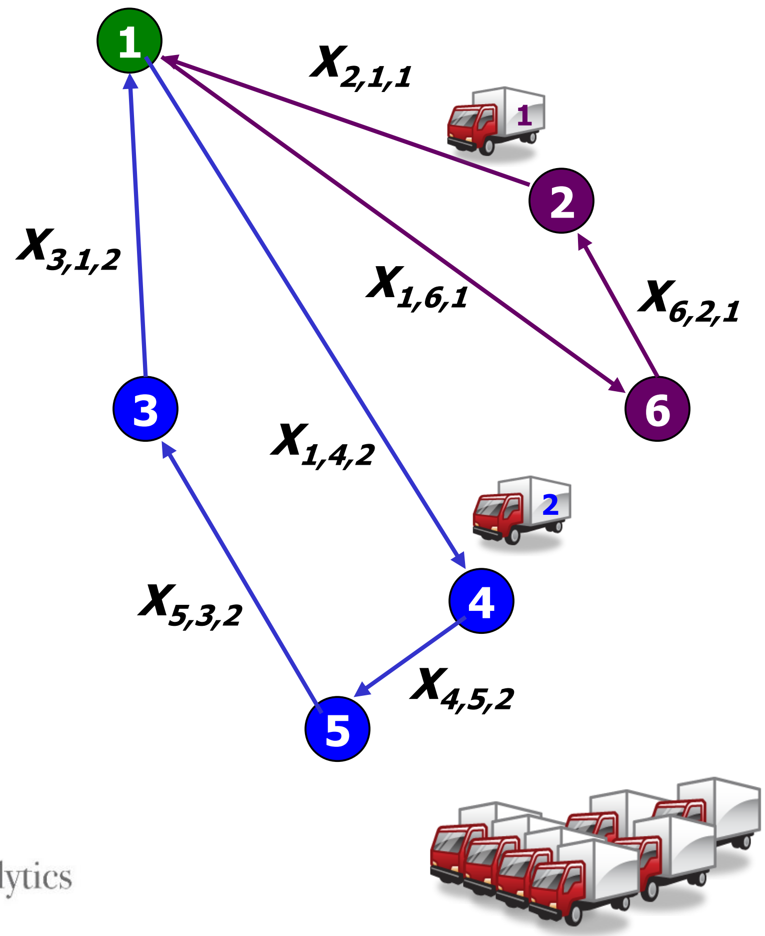


$$\textbf{Min}\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv} + \Sigma_v\, d_v y_v$$

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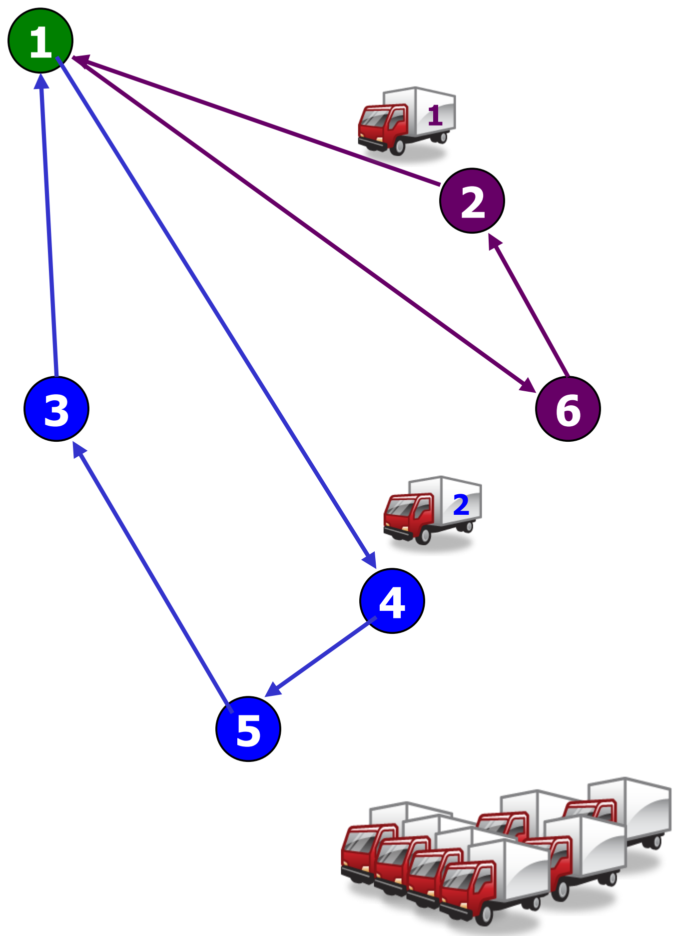
$d_v$ — Cost activate route v

$c_{ijv}$ — Cost of going from i to j using the route v

$y_v$ — **Decision to activate the route v**

$x_{ijrv}$ — Decision to go from i to j using route v

# VRP: VEHICLE ROUTING PROBLEM




$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv}\ +\ \Sigma_v\, d_v\, y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v\, x_{ijv} = 1 \quad \forall i \neq 1$$

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$$y_v \in \{0,1\},\ x_{ijv} \in \{0,1\}$$

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$d_v$ ***Cost activate route v***

$c_{ijv}$ ***Cost of going from i to j using the route v***

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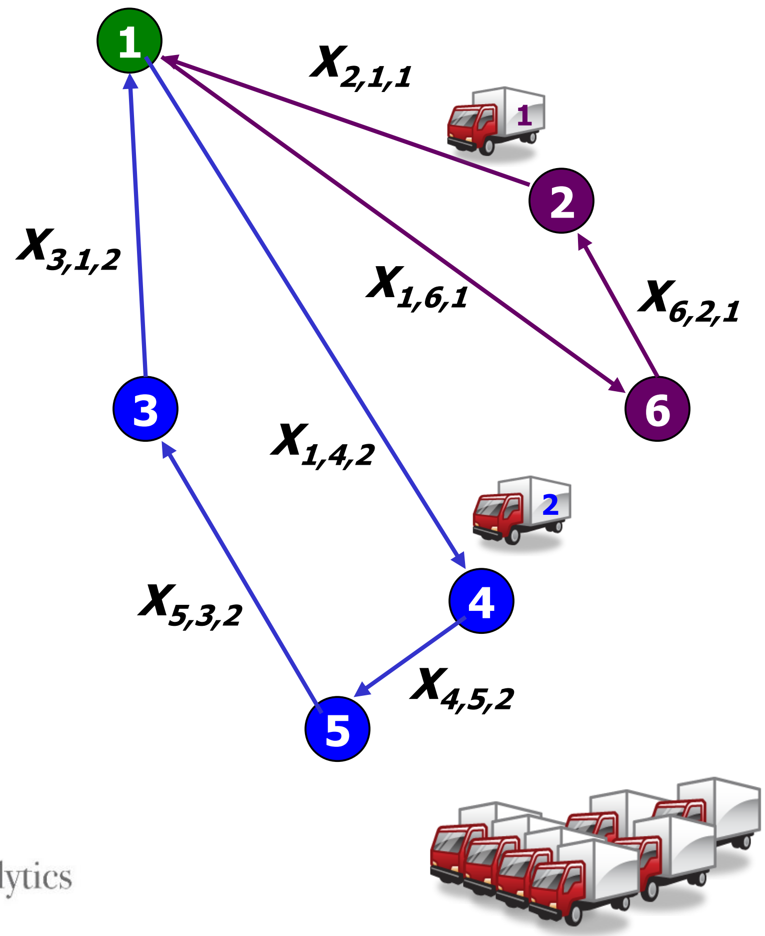
$x_{ijrv}$ ***Decision to go from i to j using route v***

# VRP: VEHICLE ROUTING PROBLEM




$$Min \ \Sigma_i \Sigma_j \ c_{ij} x_{ijv} + \Sigma_v d_v y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v x_{ijv} = 1 \quad \forall i \neq 1$$

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# VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS

$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv} + \Sigma_v\, d_v y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v\, x_{ijv} = 1 \quad \forall i \neq 1$$

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$$\Sigma_i \Sigma_j\, x_{ijv} \leq y_v \quad \forall v$$

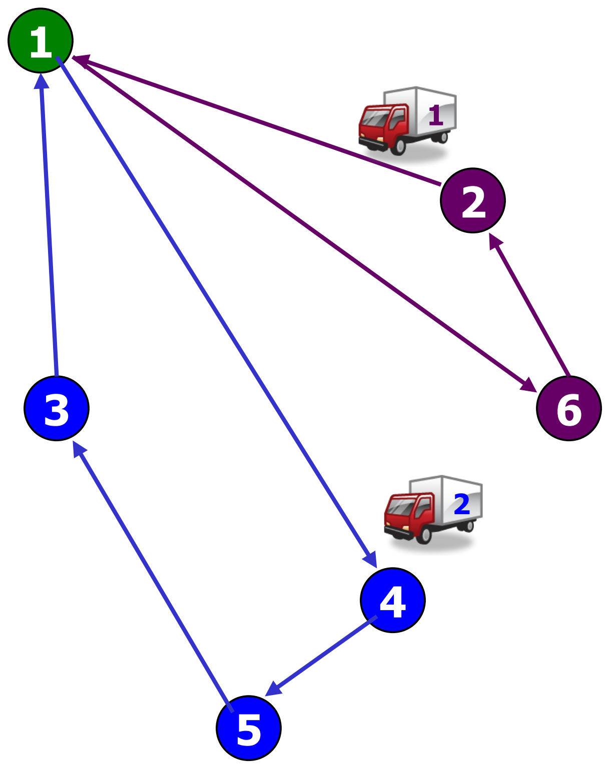
$$y_v \in \{0,1\},\ x_{ijv} \in \{0,1\}$$

$$x_{iiv} = 0$$

$$\Sigma_i \Sigma_j\ h_{ijv}\, x_{ijr} \leq Time_v \quad \forall v$$

$$\Sigma_i \Sigma_j\ v_i\, x_{ijr} \leq Volumen_v \quad \forall v$$

$$\Sigma_i \Sigma_j\ p_i\, x_{ijr} \leq Weigth_v \quad \forall v$$




| | |
|---|---|
| $h_{ijv}$ | Travel time from i to j on route v (hr) |
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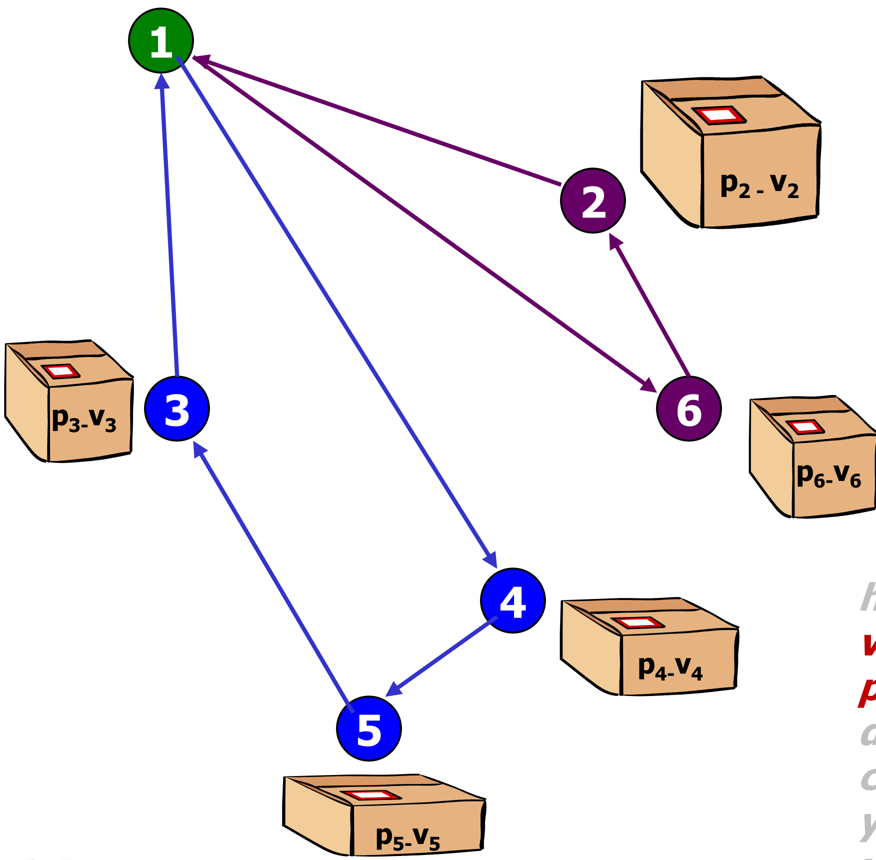
# VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS




$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv} + \Sigma_v\, d_v y_v$$

Subjet to

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$$y_v \in \{0,1\},\ x_{ijv} \in \{0,1\}$$

$$x_{iiv} = 0$$

$$\Sigma_i \Sigma_j\ h_{ijv}\, x_{ijr} \leq Time_v \quad \forall v$$

$$\Sigma_i \Sigma_j\ v_i\, x_{ijr} \leq Volumen_v \quad \forall v$$

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$h_{ijv}$ Travel time from i to j on route v (hr)
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# VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS

## SYSTEM INFORMATION APPROACH

$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv}\ +\Sigma_v\, d_v\, y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v\, x_{ijv} = 1 \quad \forall i \neq 1$$

$$\Sigma_j\, x_{jiv} = \Sigma_j\, x_{jiv} \quad \forall i \quad \forall v$$

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***MATHEMATICAL ELEMENTS/OBJECTS***

***INDEX***

***SET***

***PARAMETER***

***VARIABLE***

***CONSTRAINT***

***OBJECTIVE FUNCTION***

***PROBLEM***

OPTEX Mathematical Modeling System

# VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS
## SYSTEM INFORMATION APPROACH

$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv} + \Sigma_v\, d_v\, y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v\, x_{ijv} = 1 \quad \forall i \neq 1$$

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$$\Sigma_i \Sigma_j\ h_{ijv}\, x_{ijr} \leq Time_v \quad \forall v$$

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$$\Sigma_i \Sigma_j\ p_i\, x_{ijr} \leq Weigth_v \quad \forall v$$

**MATHEMATICAL ELEMENTS/OBJECTS**

**INDEXES:**

- *i* Node/Client
- *j* Node/Client
- *v* Route/Path/Vehicle

Analytics

# VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS
## SYSTEM INFORMATION APPROACH

$$\text{Min } \Sigma_i \Sigma_j \; c_{ij} x_{ijv} + \Sigma_v d_v y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v x_{ijv} = 1 \quad \forall i \neq 1$$

$$\Sigma_j x_{jiv} = \Sigma_j x_{jiv} \quad \forall i \;\; \forall v$$

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$$y_v \in \{0,1\}, \; x_{ijv} \in \{0,1\}$$

$$x_{iiv} = 0$$

$$\Sigma_i \Sigma_j \; h_{ijv} x_{ijr} \leq Time_v \quad \forall v$$

$$\Sigma_i \Sigma_j \; v_i x_{ijr} \leq Volumen_v \quad \forall v$$

$$\Sigma_i \Sigma_j \; p_i x_{ijr} \leq Weigth_v \quad \forall v$$

**MATHEMATICAL ELEMENTS/OBJECTS**

**SETS:**

**IMPLICIT:**

- *i* All Nodes/Clients
- *j* All Nodes /Clients
- *v* All Routes/Paths/Vehicles

**EXPLICIT:**

- $\forall i$ All Nodes/Clients
- $\forall j$ All Nodes /Clients
- $\forall v$ All Routes/Paths/Vehicles

$\forall i \neq 1$ All nodes except de "default node" warehouse

# VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS
## SYSTEM INFORMATION APPROACH

$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv} + \Sigma_v\, d_v\, y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v\, x_{ijv} = 1 \quad \forall i \neq 1$$

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$$\Sigma_i \Sigma_j\ h_{ijv}\, x_{ijr} \leq Time_v \quad \forall v$$

$$\Sigma_i \Sigma_j\ v_i\, x_{ijr} \leq Volumen_v \quad \forall v$$

$$\Sigma_i \Sigma_j\ p_i\, x_{ijr} \leq Weigth_v \quad \forall v$$

### MATHEMATICAL ELEMENTS/OBJECTS

**PARAMETERS:**

- $h_{ijv}$ Travel time from i to j on route v (hr)
- $v_i$ Weight associated with the order in i (kg)
- $p_i$ Volume associated with the order in i (m3)
- $d_v$ Cost activate route v ($)
- $c_{ijv}$ Cost of going from i to j using the route v ($)

- $Time_v$ Time available for route v
- $Volumen_v$ Volume capacity of route v
- $Weigth_v$ Weight capacity of route v

## VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS

## SYSTEM INFORMATION APPROACH

$$Min\ \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv} + \Sigma_v\, d_v\, y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v\, x_{ijv} = 1 \quad \forall i \neq 1$$

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$$\Sigma_i \Sigma_j\ h_{ijv}\, x_{ijr} \leq Time_v \quad \forall v$$

$$\Sigma_i \Sigma_j\ v_i\, x_{ijr} \leq Volumen_v \quad \forall v$$

$$\Sigma_i \Sigma_j\ p_i\, x_{ijr} \leq Weigth_v \quad \forall v$$

**MATHEMATICAL ELEMENTS/OBJECTS**

**VARIABLES:**

$y_v$ Decision to activate the route v (binary)

$x_{ijrv}$ Decision to go from i to j using route v (binary)

The variables are restricted by its type and its existence conditions

Analytics

# VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS
## SYSTEM INFORMATION APPROACH

$$Min \; \Sigma_i \Sigma_j \; c_{ij} \, x_{ijv} + \Sigma_v \, d_v \, y_v$$

**Subjet to**

$CO1_i$ $\quad \Sigma_j \Sigma_v \, x_{ijv} = 1 \quad \forall i \neq 1$

$CO2_{iv}$ $\quad \Sigma_j \, x_{jiv} = \Sigma_j \, x_{jiv} \quad \forall i \;\; \forall v$

$CO3_v$ $\quad \Sigma_i \Sigma_j \, x_{ijv} \leq y_v \quad \forall v$

$$y_v \in \{0,1\}, \; x_{ijv} \in \{0,1\}$$

$$x_{iiv} = 0$$

$TIM_v$ $\quad \Sigma_i \Sigma_j \; h_{ijv} \, x_{ijr} \leq Time_v \quad \forall v$

$VOL_v$ $\quad \Sigma_i \Sigma_j \; v_i \, x_{ijr} \leq Volumen_v \quad \forall v$

$WEI_v$ $\quad \Sigma_i \Sigma_j \; p_i \, x_{ijr} \leq Weigth_v \quad \forall v$

***MATHEMATICAL ELEMENTS/OBJECTS***

***CONSTRAINTS:***

| | |
|---|---|
| $CO1_i$ | *All nodes must visit once* |
| $CO2_{iv}$ | *If one route arrive to the node i must leave from this node.* |
| $CO3_v$ | *Only if the route is activated can visit a node* |
| $TIM_v$ | *The sum of the travel times must be less than permitted time for the route (hr)* |
| $VOL_v$ | *The sum of the volumes transported must be less than the volume capacity of the route (m3)* |
| $WEI_v$ | *The sum of the weights transported must be less than the weight capacity of the route (Kg)* |

*The constraint must satisfies its existence conditions*

# VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS
## SYSTEM INFORMATION APPROACH

$$\text{Min } cfv = \Sigma_i \Sigma_j \; c_{ij} \, x_{ijv} + \Sigma_v \, d_v \, y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v \, x_{ijv} = 1 \quad \forall i \neq 1$$

$$\Sigma_j \, x_{jiv} = \Sigma_j \, x_{jiv} \quad \forall i \quad \forall v$$

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***MATHEMATICAL ELEMENTS/OBJECTS***

***OBJETIVE FUNCTION:***

***cfv*** ***Sum of fixed and variables cost ($)***

# VRP: VEHICLE ROUTING PROBLEM WITH RESOURCES CONSTRAINTS
## SYSTEM INFORMATION APPROACH

**VRPTVW:= {**

$$Min\ \ cfv = \Sigma_i \Sigma_j\ c_{ij}\, x_{ijv} + \Sigma_v\, d_v\, y_v$$

**Subjet to**

$$\Sigma_j \Sigma_v\, x_{ijv} = 1 \quad \forall i \neq 1$$

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**}**

**MATHEMATICAL ELEMENTS/OBJECTS**

**PROBLEM:**

**VRPTVW** Vehicle Routing problem with constraint of time, volume and weight.

# An Inventory Routing Problem

An Inventory Routing Problem

A. Campbell
L. Clarke
M. Savelsbergh

April 28, 1999

http://www.doanalytics.net/Documents/An-Inventory-Routing-Problem.pdf

# 1 Introduction

PRAXAIR is a large industrial gases company with about 60 production facilities and over 10,000 customers across North America. PRAXAIR recently negotiated a policy with its customers in which PRAXAIR is in charge of managing its customers' inventories. Customers will no longer be calling PRAXAIR to request a delivery. Instead, PRAXAIR will determine who receives a delivery each day and what the size of that delivery will be. PRAXAIR will use gauge readings received from remote telemetry units as well as regular customer phone calls to monitor and forecast product inventories. The distribution planning problems associated with such vendor managed resupply policies are known as inventory routing problems.

Inventory routing problems are very different from vehicle routing problems. Vehicle routing problems occur when customers place orders and the delivery company, on any given day, assigns the orders for that day to routes for trucks. In inventory routing problems, the delivery company, not the customer, decides how much to deliver to which customers each day. There are no customer orders. Instead, the delivery company operates under the restriction that its customers are not allowed to run out of product. Another difference is the planning horizon. Vehicle routing problems typically deal with a single day, with the only requirement being that all orders have to be delivered by the end of the day. Inventory routing problems deal with a longer horizon. Each day the delivery company makes decisions about which customers to visit and how much to deliver to each of them, while keeping in mind that decisions made today impact what has to be done in the future. The objective is to minimize the total cost over the planning horizon while making sure no customers run out of product. The flexibility to decide when customers receive a delivery and how large these deliveries will be may significantly reduce distribution costs. However, this flexibility also makes it very difficult to determine a good, much less an optimal, cost effective distribution plan. When the choice becomes which of the customers to serve each day (PRAXAIR has over 10,000 customers) and how much to deliver to them, the choices become virtually endless.

Vendor managed resupply policies can be used in many situations. In some instances, the use of such a policy is natural, such as when the "customers" are really part of the same company. In others, the use of a vendor managed resupply policy is often the result of lengthy negotiations with customers who have for years followed a policy in which they call in their orders. Examples of industries where vendor managed resupply policies are being used or considered include, the petrochemical industry (gas stations), the grocery industry (supermarkets), the soft drink industry (vending machines), and the automotive industry (parts distribution). The number of industries using vendor managed resupply policies is increasing rapidly. An important reason for this is technology. For a variety of industries/products, the monitoring technology that existed several years ago was not sophisticated enough to make a vendor managed resupply system possible. The only way to check a customer's inventory for many types of products has been for the vendor to call the customer and for the customer to go look at the meter on the tank, to count the number of items in the vending machine, etc. Now the use of remote telemetry units, scanners, computers and modems allows the monitoring of inventory levels directly by the vendor, opening up new opportunities for vendor managed resupply policies.

# 2 Problem Definition

The inventory routing problem (IRP) is concerned with the repeated distribution of a single product from a single facility, to a set $N$ of customers over a planning horizon of length $T$, possibly infinity. Customer $i$ consumes the product at a rate $u_i$ (volume per day) and has the capability to maintain a local inventory of the product up to a maximum of $C_i$. The inventory at customer $i$ is $I_i^0$ at time 0. A fleet $M$ of homogeneous vehicles, with capacity $Q$, is available for the distribution of the product. The objective is to minimize the average daily distribution cost during the planning period without causing stockouts at any of the customers. Vehicles are allowed to make multiple trips per day. Three decisions have to be made:

1. When to serve a customer?

2. How much to deliver to a customer when served?

3. Which delivery routes to use?

Real-life inventory routing problems are obviously stochastic. No customer will use product the same way every single day. In many situations, however, usage is relatively predictable and customers generally use about the same amount each day if we look at their total usage for several days in a row. Therefore, solution approaches developed for the IRP as defined above will still provide useful planning tools.

# 3 Solution Approach

Even though the IRP is a long-term problem, almost all proposed solution approaches solve only a short-term version of the problem to make it easier. In early work, short-term was often just a single day, but in later work this was expanded to several days. Besides the number of days modeled, key features that distinguish different solution approaches include how the long-term effect of short-term decisions is modeled, and how it is determined which customers are included in the short-term problem. For a review of some of these approaches, we refer to Campbell et al. [2].

A short-term approach has the tendency to defer as many deliveries as possible to the next planning period, which may lead to an undesirable situation in the next planning period. Therefore, the proper projection of a long-term objective into a short-term planning problem is essential. It needs to capture the costs and benefits of delivering to a customer earlier than necessary. Our focus has been on developing a flexible system capable of handling large instances that properly balances short-term and long-term goals and that considers all the key factors, i.e., geography, inventory, capacity, and usage rate. We wanted also to create a system that would consider routing customers together on a day where none of them are at the point of run out, but where they combine to make a good full truckload delivery. We found that most systems reduce the problem by starting with only the "emergency" customers, never putting together certain combinations that make sense with regard to location and delivery size. The basis for our system is a two-phase solution approach. In the first phase, we determine which customers receive a delivery on each day of the planning period and decide on the size of the deliveries. In the second phase, we determine the actual delivery routes and schedules for each of the days.

### 3.1 Phase I: An Integer Programming Model

At the heart of the first phase is an integer program. Central to the model are two quantities: $L_i^t = \max(0, tu_i - I_i^0)$, a lower bound on the total volume that has to be delivered to customer $i$ by the end of day $t$, and $U_i^t = tu_i + C_i - I_i^0$, an upper bound on the total volume that can be delivered to customer $i$ by the end of day $t$. Let $d_i^t$ represent the delivery volume to customer $i$ on day $t$, then to ensure that no stockout occurs at customer $i$ and to ensure that we do not exceed the inventory capacity at customer $i$, we need to have that

$$L_i^t \leq \sum_{1 \leq s \leq t} d_i^s \leq U_i^t \quad \forall i \in N,\ t = 1, ..., T.$$

$$L_i^t = \max(0, tu_i - I_i^0) \qquad\qquad U_i^t = tu_i + C_i - I_i^0$$

To model the resource constraints with some degree of accuracy and to have a meaningful objective function, we found it necessary to explicitly use delivery routes. However, when we refer to a "route", we are really referring to a set of customers without enforcing a specific ordering among the customers in the set. We estimate the distance required to visit the customers in the set by the length of the optimal traveling salesman tour through all the customers. Now, let $R$ be the set of delivery routes, let $T_r$ denote the duration of route $r$ (as a fraction of a day), and let $c_r$ be the cost of executing route $r$. Furthermore, let $x_r^t$ be a 0-1 variable indicating whether route $r$ is used on day $t$ ($x_r^t = 1$) or not ($x_r^t = 0$). The total volume that can be delivered on a single day is limited by a combination of capacity and time constraints. Since vehicles are allowed to make multiple trips per day, we cannot simply limit the total volume delivered on a given day to be the sum of the vehicle capacities.

To be more precise, the resource constraints can be modeled by

$$\sum_{i:i\in r} d_{ir}^t \leq Q x_r^t \quad \forall r \in R,\ t = 1, ..., T,$$

and

$$\sum_{r:r\in R} T_r x_r^t \leq |M| \quad t = 1, ..., T.$$

These constraints ensure that we do not exceed the vehicle capacity on any of the selected routes and that the time required to execute the selected routes does not exceed the time available.

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The basic Phase I integer programming model is given by

$$\min \sum_t \sum_r c_r x_r^t$$

$$L_i^t \leq \sum_{1 \leq s \leq t} d_i^s \leq U_i^t \quad \forall i \in N,\ t = 1, ..., T,$$

$$\sum_{i: i \in r} d_{ir}^t \leq Q x_r^t \quad \forall r \in R,\ t = 1, ..., T,$$

$$\sum_{r: r \in R} T_r x_r^t \leq |M| \quad t = 1, ..., T.$$

$$x^t{}_r \in \{0,1\}$$

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OPTEX Mathematical Modeling System

The basic Phase I integer programming model is given by

$$\min \sum_t \sum_r c_r x_r^t$$

$$L^t_i \leq \sum_{r \in RR(i)} \sum_{1 \leq s \leq t} d^s_{i,r} \leq U^t_i \quad \forall i \in N,\ t = 1, \ldots, T,$$

$$\sum_{i:i \in r} d_{ir}^t \leq Q x_r^t \quad \forall r \in R,\ t = 1, \ldots, T,$$

$$\sum_{r:r \in R} T_r x_r^t \leq |M| \quad t = 1, \ldots, T.$$

$$x^t_r \in \{0,1\}$$

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The first variation of the basic model handles fixed and variable stop times at the customers as well as a vehicle reloading time at the facility. The duration of a route $T_r$ can be modified to include not only the estimated time to drive the distance between the customers on the route, but also a fixed stop time for each customer and an initial fill time for the vehicle required before the route can start. Dispense time at a customer clearly cannot be included in $T_r$ *a priori* because it depends on the size of the delivery. Therefore, we must alter the resource constraint as follows, where $F$ is the percentage of the day required to dispense each unit of product

$$\sum_{r:r \in R} (T_r x_r^t + \sum_{i:i \in r} F d_{ir}^t) \leq |M| \quad t = 1, ..., T.$$

The second variation handles operating modes of customers. Operating mode refers to the start and end time of customer usage on each day of the week. Before, we assumed that each customer $i$ uses product 24 hours per day everyday. Operating modes are important. When a customer does not use product on the weekend, for example, it has a big impact on properly timing the deliveries. Operating modes can be handled easily by appropriately modifying the lower and upper bound parameters. The values of the lower and upper bounds on day $t$ now depend on where in the week days 1 through $t$ fall.

$$L^t_i = \max\left(0, \sum_{1\le s\le t} u^s_i - I^0_i\right) \qquad U^t_i = \sum_{1\le s\le t} u^s_i + C_i - I^0_i$$

The third variant handles time windows at customers. An operating mode restricts when a customer uses product. A time window restricts when a customer can receive a delivery. Time windows may be day dependent as well. To handle time windows, the lower and upper bound parameters need to modified again, but in a slightly different way. Now the lower bound $L_i^{\ddagger}$ needs to be defined as the total volume that has to be delivered to customer $i$ by the closing of the time window on day $t$ to allow customer $i$ to last until the opening of the time window on day $t + 1$ (or the opening of the time window on the first available day for the next delivery if no deliveries can be made on day $t + 1$). The upper bound $U_i^{\ddagger}$ is now defined as the largest volume that customer $i$ can receive by the close of the delivery window on day $t$.

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$$\mathbf{d^t_i \leq q^t_i}$$

$\mathbf{q^t_i}$ cantidad de producto que puede recibir el cliente **i** en día **t**

## 3.2 Phase I: Solving the Integer Programming Model

The integer programming model presented above is not very practical for two reasons: the huge number of possible delivery routes and, although to a lesser extent, the length of the planning horizon. To make the integer program computationally tractable we consider a small (but good) set of delivery routes and aggregate periods toward the end of the planning horizon.

### 3.2.1 Clusters

Our approach to reduce the number of routes is based on allowing customers to be on a route together only if they are in the same *cluster*. A cluster is a group of customers that can be served cost effectively by a single vehicle for a long period of time. The cost of a cluster is an approximation of the distribution cost for serving the customers in the cluster for a month. The cost of serving a cluster does not only depend on the geographic locations of the customers in the cluster, but also on whether the customers in the cluster have compatible inventory capacities and usage rates. Therefore, to evaluate the cost of a cluster, we need a model that considers all of these factors.

The following approach is used to identify a good set of disjoint clusters covering all customers:

1. Generate a large set of possible clusters.

2. Estimate the cost of serving each cluster.

3. Solve a set partitioning problem to select clusters.

### 3.2.2 Aggregation/Relaxation

Given that our two-phase solution approach will be embedded in a rolling horizon framework, the emphasis should be on the quality and detail of the decisions concerning the first few days of the plan. This provides us with an excellent opportunity to reduce the size of the integer program by aggregating days towards the end of the planning period.

For the first $k$ days, we will still have route selection variables for each day, but for the days after that, we will have route selection variables covering periods of several days. Instead of making a decision on whether to execute each route on days 8 to 14 individually, for example, we now decide how many times each of the routes will be executed during the whole week instead. Several aggregation schemes were tested. We found that considering weeks rather than days towards the end of the planning horizon still does a good job of preserving the costs associated with the effect of short-term decisions on the future and yields a significant reduction in CPU time. Therefore, the daily variables associated with these later days are replaced by weekly variables. Upper and lower bounds are altered accordingly as well.

A further simplification is obtained by relaxing the integrality restrictions on the variables representing the weekly decisions. Therefore, the only binary variables appearing in the integer program will be those representing route selections for the first $k$ days.

## 3.3 Phase II: Scheduling

A solution to the integer program of Phase I specifies the volumes to deliver to each customer for the next $k$ days. It does not specify departure times and customer sequences for the different vehicles. Therefore, we still need to construct vehicle routes and schedules.

Since the delivery volumes specified by the solution to the integer program may not fit before a specific time of the day and may need to be received before a certain later time to prevent run out, these deliveries have self imposed time windows. Therefore, to convert the information provided by the solution to the integer program to daily vehicle routes and schedules, we can solve a sequence of vehicle routing problems with time windows.

However, such an approach does not capitalize on the flexibility inherent in the inventory routing problem. The delivery volumes specified by the solution to the integer program are good from a long-term perspective; they may not be good from a short-term perspective. Therefore, we treat the delivery volumes and timing specified by the solution to the integer programs as suggestions. We try to follow these suggestions as closely as possible, since this helps to achieve our long-term goals, but we allow small deviations when it helps to construct better short-term plans. To be more precise, we construct vehicle routes and schedules for two consecutive days, where we force the total volume delivered to a customer over the two days to be greater than or equal to the total delivery volume specified by the solution to the integer program for these two days, but we do not enforce specific delivery volumes on individual days.

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In this way, we stay close to the delivery volumes suggested by the integer program, which is good from a long-term perspective, but we introduce some flexibility in the daily routing and scheduling, which is good from a short-term perspective. Deliveries can be split into smaller pieces, delivering one part on the first day and the second part on the second day if this works out to be better, for example when resources are very tight on one of the days. This flexibility is even more important when we consider the fact that in practice a few customers may not follow a vendor managed resupply policy and may call in orders that need to be added to the daily routing and scheduling problem. With new orders and new accurate up-to-date information on customer inventory levels, it may make sense to shift around some of the deliveries over the next couple of days.

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Because of customer usage and customer inventory capacities, there may be customers that require a delivery on both days or even multiple times a day. Consequently, in our two day routing and scheduling problem, we can distinguish two types of customers: customers that require multiple deliveries over the two days and customers that require only one.

We have developed and implemented an insertion heuristic for this two day routing and scheduling problem. The heuristic is a logical progression of commonly used techniques in insertion heuristics for the vehicle routing problem with time windows, see for example Solomon [5] and Kindervater and Savelsbergh [4].

"the computer-based mathematical modeling is the greatest invention of all times"

**Herbert Simon**

First Winner of Nobel Prize in Economics (1978)

**"for his pioneering research into the decision-making process within economic organizations"**